

Ch 9 - Production function

Production function

The firm's production function for a particular good q is:

$$q = f(k, l)$$

which shows the maximum amt of the good that can be produced using alternative combinations of capital and labor.

Marginal physical product (marginal productivity)

The marginal physical product of an input is the additional output that can be produced by employing one more unit of that input while holding all other inputs constant.

Mathematically, why? usually when $l \uparrow$ ceteris paribus (normal input) $q \uparrow$

$$\text{marginal physical product of labor} = MP_L = f_L = \frac{\partial q}{\partial L}$$

$$\text{" " of capital} = MP_K = f_K = \frac{\partial q}{\partial K}$$

Note that it is 'partial derivative' to reflect the fact that all other input usage is held constant while input of interest is being varied.

Diminishing Marginal productivity:

$MP_{K \text{ or } L}$ depends on how much of that input is used. Labor for example cannot be added indefinitely to a given fixed without exhibiting some decline in its productivity.

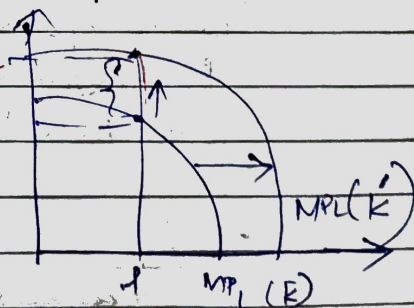
The assumption (law) of diminishing marginal productivity is an assumption that the second order partial derivatives of the production function.

$$\frac{\partial MP_K}{\partial K} = \frac{\partial^2 Q}{\partial K^2} = f_{KK} = f_{11} < 0$$

$$\frac{\partial MP_L}{\partial L} = \frac{\partial^2 Q}{\partial L^2} = f_{LL} = f_{22} < 0$$

This was originally proposed by Thomas Malthus and his production ~~and~~ economics to be called "dismal science".

BUT, the changes in the ^{of an input} marginal productivity over time ^{long run} also depends on the changes in other inputs like.



if $\frac{\partial MP_K}{\partial K}$ // also depend on $\frac{\partial MP_K}{\partial L}$.

The intuition is that

$$\frac{\partial MPL}{\partial K} = f_{LK} = f_{21} > 0$$

$$\frac{\partial MPK}{\partial L} = f_{KL} = f_{12} > 0$$

For example, the increase in capital inputs can offset the impact of diminishing marginal productivity of labor alone.

Average physical productivity.

In common usage, the term labor productivity means average productivity. When it is said that a certain industry has experienced productivity increase, this is taken to mean that output per unit of labor input has increased. It is often used as a measure of efficiency.

$$AP_L = \frac{\text{output}}{\text{labor input}} = \frac{q}{l} = \frac{f(k, l)}{l}$$

Note that AP_L also depends on the level of capital employed. (If capital $\uparrow$, output per unit of labor $\uparrow$ increase as each labor gets new machines and stuffs.)

eg 9.1

$$q \rightarrow f(k, l) = 600k^2l^2 - k^3l^3$$

To find MP_L we keep k constant

$$\begin{aligned} \frac{\partial q}{\partial l} &= 600k^2 \cdot 2l - k^3 \cdot 3l^2 \\ &= 1200k^2l - 3k^3l^2 \end{aligned}$$

Take $k = 10$

$$\Rightarrow \frac{\partial q}{\partial l} = 120,000l - 3000l^2$$

Setting $MP_L = 0$

$$\Rightarrow 120,000l - 3000l^2 = 0$$

$$-3000l + 120,000 = 0$$

$$-3000l = -120,000$$

$$l = 40$$

$$\frac{\partial^2 q}{\partial l^2} \Big|_{l=40} = 120,000 - 6000(40)$$

$$= 120,000 - 240,000$$

$$= -120,000 < 0$$

$\Rightarrow l = 40$ is the maximising unit of labor.

Note: If we $\uparrow$ l beyond 40, $q \downarrow$.

$$\text{Average product} = \frac{q}{l} \Big|_{k=10} = \frac{60,000l^2 - 1,000l^3}{l}$$

$$= 60,000l - 1000l^2$$

AP reaches max-

$$\text{when } \frac{\partial AP_L}{\partial l} = 0$$

$$\Rightarrow 60,000 - 2000l = 0$$

$$\Rightarrow 2000l = 60,000$$

$$l = 30$$

At $l = 30$
 $APL = 60,000(30) - 1,000(900)$
 $= 18,00,000 - 9,00,000$
 $= 9,00,000$

At $l = 30$
 $MP_L = 1,20,000(30) - \frac{3000(30)^2}{900}$
 $= 36,00,000 - 27,00,000$
 $= 9,00,000$

* Thus when AP_L is maximum,
 $AP_L = MP_L$

Generalising

$$APL = \frac{q}{l} = \frac{f(k, l)}{l}$$

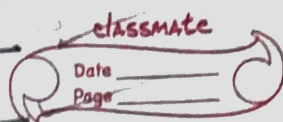
$$\frac{\partial (APL)}{\partial l} = \frac{l f'_l - f(k, l)}{l^2}$$

$$\frac{\partial (APL)}{\partial l} = \frac{l \cdot MP_L - q}{l^2}$$

$$K=11 \Rightarrow Q = 72,600L^2 - 1331P^3$$

$$MP_L = 1,45,200L - 3993P^3$$

$$AP_L = \frac{Q}{L} = 72,600 - 1331P^3$$



When AP_L is maximum

$$\frac{\partial AP_L}{\partial L} = 0$$

$$\Rightarrow \frac{L \cdot MP_L - Q}{L^2} = 0$$

$$\Rightarrow L \cdot MP_L - Q = 0$$

$$\Rightarrow L \cdot MP_L = Q$$

$$\Rightarrow MP_L = \frac{Q}{L}$$

$$\Rightarrow \boxed{MP_L = AP_L}$$

Thus when AP_L is maximum, $MP_L = AP_L$

Coming back to the numerical example we can see the Q is greatest at $L=40$ than $L=30$ but output per worker is highest at 30. Why? Because in this example we kept K const at $K=10$, thus the diminishing MP_L eventually results in a declining level of output per worker.

How would an $\uparrow$ in K affect MP_L and AP_L ?
Let's see.

as $K \uparrow$, each unit of labor can produce more and thus with an add'l unit of L , more is produced thus MP_L increases. The average productivity also increase as fixed no of machinery makes each labor to produce more and hence output produced per worker $\uparrow$.

Isocuant maps and the rate of technical substitution

To illustrate possible substitution of one input for another in a production function, we use its isocuant map.

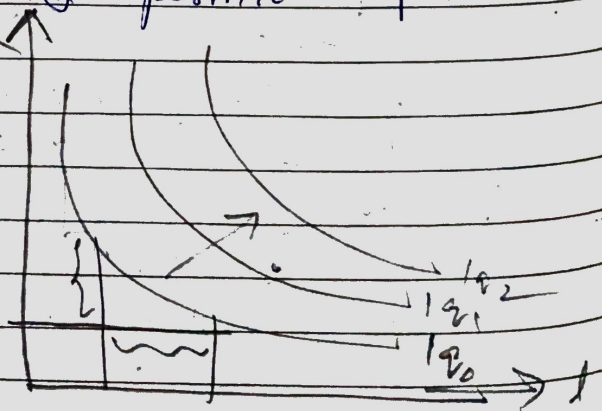
Isocuant is a locus of all alternative combinations of inputs that can be used to produce a given level of output.
i.e. $f(x_1, x_2) = q_0$.

The slope of these curves show the rate at which x_1 can be substituted for x_2 while keeping output constant.

The slope is called (marginal) rate of technical substitution (RTS). Its value is diminishing for increasing inputs of labor.

Each isocuant represents a different level of output. They rise up successively higher levels of output as we move in a north easterly direction.

Presumably, using more of each of the inputs will permit output to increase.



Marginal rate of technical substitution

(MRTS shows the rate at which one input (say labor) can be substituted for another input (say capital) while holding output constant along an isoquant.

In mathematical terms,

$$RTS(l \text{ for } k) = \left. \frac{-dk}{dl} \right|_{q=q_0}$$

RTS and Marginal productivities

The RTS (of l for k) is equal to the ratio of MP_l to MP_k i.e.

$$\left. \frac{-dk}{dl} \right|_{q=q_0} = \frac{MP_l}{MP_k}$$

not $\Rightarrow$

$$dq = \frac{\partial f}{\partial l} \cdot dl + \frac{\partial f}{\partial k} \cdot dk \quad \frac{-dk}{dl} = \frac{MP_l}{MP_k}$$

$$= MP_l \cdot dl + MP_k \cdot dk$$

Along an isoquant $dq = 0$ (output is constant)

$$\Rightarrow 0 = MP_l \cdot dl + MP_k \cdot dk$$

$$\Rightarrow MP_l \cdot dl = -MP_k \cdot dk$$

This says that along an isoquant, the gain in output from $\uparrow$ of l slightly is exactly balanced by the loss in output from suitably decrease k.

$$\text{Rearranging} \quad \left. \frac{-dk}{dl} \right|_{q=q_0} = RTS(l \text{ for } k) = \frac{MP_l}{MP_k}$$

This eqn shows that there is a point that we actually observe must be -ve. Because both MP_L and MP_K will be non -ve as no firm would bear the cost to employ an additional unit to quit to reduce the output. ~~Also~~ eventually meaning RTS is the neg slope of isoquant is the -ve of RTS .

Reasons for a diminishing RTS

Along an isoquant, RTS diminishes. we have seen the isoquants are not only downward sloping but are convex.

→ as $L \uparrow \rightarrow RTS \downarrow$ along isoquant.

~~as $\frac{K}{L} \downarrow$~~ $\downarrow RTS \downarrow$ along an isoquant

One might think a rise in L along with a fall in K result in $\downarrow$ in MP_L and $\uparrow$ in MP_K and thus a fall in RTS. But it's not the case. It is not possible to derive a diminishing RTS from the assumption of diminishing productivity alone.

For isoquants are convex, we must show
 $\frac{dRTS}{dl} < 0$

Now assume $q = f(k, l)$
 $f_k > 0$ $f_l > 0$
 $f_{kk} < 0$ $f_{ll} < 0$

$$\frac{dRTS}{dl} = \frac{d(f_l/f_k)}{dl}$$

$$= \frac{f_k \left(f_{ll} + f_{lk} \frac{dk}{dl} \right) - f_l \left(f_{kl} + f_{kk} \frac{dk}{dl} \right)}{(f_k)^2}$$

Given $\frac{dk}{dl} = -\frac{f_l}{f_k}$ and along an isoquant
 and Y.T $\Rightarrow f_{kl} = f_{lk}$

$$\text{We have } \frac{dRTS}{dl} = \frac{f_k \left(f_{ll} + f_{lk} \left(-\frac{f_l}{f_k} \right) \right) - f_l \left(f_{lk} + f_{kk} \left(-\frac{f_l}{f_k} \right) \right)}{(f_k)^2}$$

$$= \cancel{f_k f_{ll}} + \cancel{f_{lk} f_l} - \cancel{f_l f_{lk}} + \cancel{f_l^2 f_{kk}} = f_l^2 f_{kk}$$

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-ve -ve iff $f_{KK} > 0$ -ve

$$= f_K^2 f_{LL} - 2 f_K f_L f_{KL} + f_L^2 f_{KK}$$

$$(f_K)^3$$

$$f_K > 0 \Rightarrow (f_K)^3 > 0$$

$$f_L > 0 \Rightarrow f_L^2 > 0$$

$$f_K > 0 \Rightarrow f_K^2 > 0$$

$$f_{KK} < 0 \Rightarrow f_L^2 f_{KK} < 0$$

$$f_{KL} < 0 \Rightarrow f_K^2 f_{KL} < 0$$

$$f_K > 0, f_L > 0 \Rightarrow -2 f_K f_L < 0$$

$$-2 f_K f_L f_{KL} < 0 \text{ iff } f_{KL} > 0$$

If so then the whole expression

$$\frac{MRTS}{dI} < 0$$

(They are strictly quasi concave fns)

Importance of cross productivity effects (f_{KL} & f_{LK}).

Convexity stays only when cross productivity is positive.

Intuitively it makes sense as suppose if workers get more capital, their productivity increases.

But there are certain production fns where $f_{KL} = f_{LK} < 0$ for at least some range of input values.

Thus when we assume diminishing MRTS, we are not simply assuming diminishing marginal

productivities but we are specifically assuming that the marginal product diminishes so rapidly enough to compensate for any possible negative cross productivity effects.

eg 9.2:

$$q = f(k, l) = 600k^2l^2 - k^3l^3$$

$$MR_L = f_L = \frac{\partial q}{\partial l} = 1200k^2l - 3k^3l^2$$

$$MR_K = f_K = \frac{\partial q}{\partial k} = 1200kl^2 - 3k^2l^3$$

$$1200k^2l - 3k^3l^2 > 0$$

$$400k^2l - k^3l^2 > 0$$

$$400 - kl > 0$$

$$kl < 400$$

ie for $MP_L > 0$ iff $kl < 400$

$$f_{LL} = \frac{\partial^2 q}{\partial l^2} = 1200k^2 - 6k^3l$$

$$f_{KK} = \frac{\partial^2 q}{\partial k^2} = 1200l^2 - 6kl^3$$

iii) $f_{LL}, f_{KK} < 0$ iff $kl > 200$

Thus the marginal productivity relations for this fn behave normal within the range $200 < kl < 400$.

We have seen earlier that cross partials
 is cross productivity being positive & very
 crucial for the normalness of very
 MRTS and the convexity -

$$f_{KL} = f_{LK} = 2400 K l - 9 K^2 l^2$$

$$2400 K l - 9 K^2 l^2 > 0$$

$$266.6 - K l > 0$$

$$K l < 266.6$$

$\therefore \frac{dRTS}{dl}$ // only be -ve iff

$$200 < K l < 266.6$$

Returning to see

if $K=1$

$$MP_L = 1200 l^3 - 3 l^5$$

$$MP_{LK} = 1200 l^3 - 3 l^5$$

$$f_{ll} = f_{KK} = 1200 l^2 - 6 l^4$$

when $K=1$ and $f_{ll} = f_{KK}$ for this production fn'

$$\frac{dRTS}{dl} = \frac{f_K^2 f_{ll} - 2 f_K f_l f_{Kl} - f_l^2 f_{KK}}{(f_K)^3}$$

$$= \frac{f_K^2 f_{ll} - 2 f_l f_l f_{Kl} - f_l^2 f_{KK}}{(f_l)^3}$$

A local measure of RTS is $e_{q,c} = \frac{\partial f(x, y)}{\partial c}$ is a scale elasticity.

$$= \frac{f_1^2 f_{11} - 2 f_1 f_{12} f_{21} + f_2^2 f_{22}}{(f_1)^3}$$

$$= \frac{-2 f_1^2 f_{12}}{(f_1)^3}$$

$$= \frac{-2 f_{12}}{f_1}$$

$$= -2 \frac{f_{12}}{f_1} \xrightarrow{\text{ne}} -ve < 0$$

$f_1 > 0$

The numerator 'd' simplify to $-2 f_1^2 f_{12} = -2 f_{12} f_1^2$
 ($f_1 > 0$ denominator is +ve)
 $\xrightarrow{-ve - +ve} -ve \Rightarrow \text{Convexity}$

Returns to Scale

The concept of homogeneous function

A function is said of 2 variables x and y defined in a domain D is said to be homogeneous of degree k , $y \neq (x, y)$ in D ,
 $f(tx, ty) = t^k f(x, y) \forall t > 0$

$$x \cdot f'_x + y \cdot f'_y = k \cdot f(x, y)$$

$\downarrow$
degree

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$f(x, y)$ is homogeneous of degree $k \Leftrightarrow$

$$x \cdot \frac{\partial f}{\partial x} + y \cdot \frac{\partial f}{\partial y} = k f(x, y).$$

$$f(tx, ty) = t^k f(x, y)$$

$$f'_x \frac{dtx}{dt} + f'_y \frac{dty}{dt} = k \cdot t^{k-1} f(x, y)$$

If $z = f(x, y)$ is homogeneous of degree k ,
 then its first partials f'_x and f'_y will
 always be homogeneous with degree $k-1$.

$$f'(tx, ty) = t^{k-1} f'(x, y)$$

$$f''_{xx}(tx, ty) = t^{k-2} f''_{xx}(x, y)$$

$$f''_{xx} \cdot x^2 + 2f''_{xy} \cdot xy + f''_{yy} \cdot y^2 = (k-1)k f(x, y).$$

Intuition behind returns to scale:

if inputs are doubled

→ more efficiency as greater division
 of labor & specialization of L
 OR

→ Entailing some costs because
 managerial overhead may be
 difficult.

Returns to scale

If the production function is given by $q = f(k, l)$ and if all inputs are multiplied by the same the constant t (where $t > 1$), then we classify the returns to scale of the prodn by

Effect on output	Returns to scale
(1) $f(tk, tl) = t f(k, l) = tq$	constant
(2) $f(tk, tl) < t f(k, l) = tq$	Decreasing
(3) $f(tk, tl) > t f(k, l) = tq$	Increasing

- (1) - proportionate (2) less than proportionate (3) more than proportionate
Change in outcome as inputs were changed by same factor $t, t > 1$.

Constant Returns to Scale

There are economic reasons for why a firm's production function might exhibit constant returns to scale. If the firm operates many identical plants, it may ↑/↓ production simply by varying the no of them in current operation.

When it's CRS, it's homogenous.

$$f(tk, tl) = t^1 f(k, l)$$

We already know that if a fn is homo of degree α , it's first partials are homo of $\alpha - 1$ degree.

$$MP_K = \frac{\partial f(k, l)}{\partial k} = \frac{\partial f(tk, tl)}{\partial (tk)} \quad \text{In general } \frac{\partial f(n)}{\partial x_i} = \frac{\partial f(n)}{\partial x_i} \cdot \frac{1}{n}$$

$$MP_L = \frac{\partial f}{\partial l} = \frac{\partial f(tk, tl)}{\partial (tl)} \quad \frac{\partial f(n)}{\partial x_i} = \frac{\partial f(n)}{\partial x_i} \cdot \frac{1}{n}$$

in particular let $\frac{t}{1} = 1$

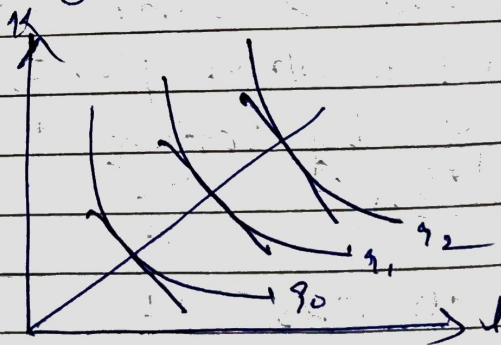
$$\Rightarrow MP_L = \frac{\partial f(K/L, 1)}{\partial L}$$

$$MP_L = \frac{\partial f(K/L, 1)}{\partial L}$$

i.e. MP of any input depends only on the ratio of capital to labor input, not on the absolute levels of these inputs.

Homothetic prod. fn.

For CRS, RTS depends only on the ratio of K to L , not on the scale of production. Each isoquant will be radial expansions of one another. Along any ray through origin, where the ratio K/L doesn't change, the slopes of successively higher isoquants are identical.



on CRS case, expanding the level of production doesn't alter trade-off of any inputs, so prod. fns are homothetic!

A production function can be a homothetic indifference curve, even if it doesn't exhibit CRS. It can be returned by an appropriate transformation of a homofun. Hence IRS/DRS can be incorporated into a CRS through appropriate transformation like that of exponential.

$$F(k, l) = [f(k, l)]^{\gamma} \quad \gamma > 1$$

$$F(tk, tl) = [f(tk, tl)]^{\gamma}$$

$$= t^{\gamma} [f(k, l)]^{\gamma}$$

$$= t^{\gamma} F(k, l) \Rightarrow t F(k, l) \quad \text{IRS}$$

||| DRS with $\gamma < 1$
Thus IRS / DRS can be incorporated.

The n input case

$$q = f(x_1, x_2, \dots, x_n)$$

If all inputs are scaled by $t > 1$, we have

$$f(tx_1, tx_2, \dots, tx_n) = t^k f(x_1, x_2, \dots, x_n) = t^k q$$

$k > 1 \Rightarrow$ CRS $k < 1 \Rightarrow$ DRS $k > 1$ - IRS

The usual point is that all inputs are scaled by the same amount. But this might not be possible in all real-world scenarios (read more on pg 293).

Elasticity of Substitution

EOS determines the shape of a single isoquant rather than the overall isoquant map.

We know $RTS \downarrow$ as $K/L \downarrow$.

Now EOS measures the degree of change of RTS as K/L changes.

If RTS doesn't change at all as K/L changes it means substitution is easy as the ratio of the marginal product of the 2 inputs doesn't change as input mix changes.

If RTS changes rapidly for small changes in K/L , we would say that substitution is difficult because minor variations in the input mix will have substantial effect on the inputs' relative productivities.

For the prod fn $q = f(K, L)$ the EOS (σ) measures the proportional change in K relative to the proportional change in the RTS along an isoquant.

$$\sigma = \frac{\% \Delta (K/L)}{\% \Delta (RTS)} = \frac{d(K/L)}{dRTS} \cdot \frac{RTS}{K/L}$$

$$= \frac{\partial \ln(K/L)}{\partial \ln RTS} = \frac{\partial \ln K/L}{\partial \ln f_L/f_K}$$

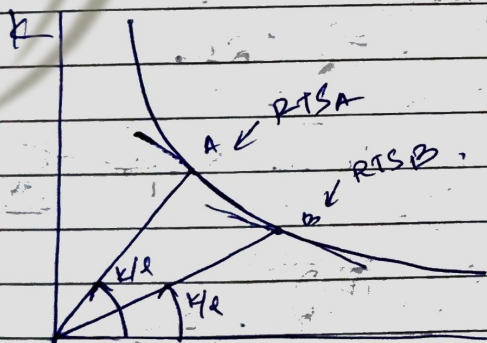
The value of σ is always true as along an isoquant k and RTS move in the same direction.

If σ is high RTS will not change much relative to k/l and l/c will be flat.

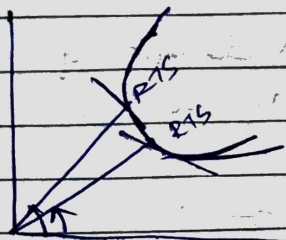
A low value of $\sigma \Rightarrow$ a rather sharply curved isoquant. It is relatively the RTS will change by a substantial amount as k/l changes.

Note:

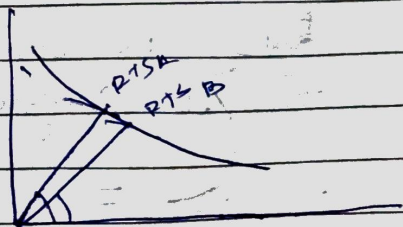
It is possible that σ will vary as one moves along an isoq. and as the scale of prod changes. Often, however, it is convenient to assume that σ is constant along an isoq. If the prod fn is also homothetic, then all isoq are radial expansions $\Rightarrow \sigma$ will be same for all.



AS $k/l \downarrow$ RTS $\downarrow$ from A to B.



σ low
RTS changes rapidly for a small change in k/l



σ high
RTS doesn't change much for a change in k/l .

the n input one.

$$\sigma_j = \frac{\partial \ln(\pi_i / \pi_j)}{\partial \ln(P_i / P_j)}$$

q is const
all other inputs const

(Read more on pg 295):

Cases

(1) Linear ($\alpha = \infty$)

$$q = f(K, L) = aK + bL$$

We can see

$$\begin{aligned} f(tK, tL) &= atK + btL \\ &= t(aK + bL) \\ &= t f(K, L) \end{aligned}$$

$\Rightarrow$ CRS

All lqs are || straight lines with slope = $-\frac{b}{a}$

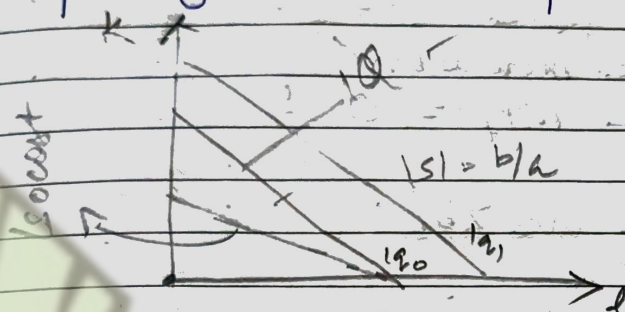
We know

$$\alpha = \frac{\% \Delta \ln K/L}{\% \Delta \ln RTS} = \frac{d(K/L)}{dRTS} \cdot \frac{RTS}{K/L}$$

RTS is constant along a straight lq
 $\Rightarrow$ no change in RTS along the lq

$$\alpha = \frac{\% \Delta \ln K/L}{0} = \infty$$

Here α is very high RTS doesn't change much for change in $\frac{K}{L}$. They can be thought of as perfect substitutes. An industry characterized by such a production function could use only capital or only labor depending on these inputs price (as of the boundary).



Case (2) Fixed proportions ($\alpha = 0$).

The production function is the case of fixed proportions production function where capital and labor must always be used in a fixed ratio. The graph will be L-shaped and will always operate along the ray where k is constant (kinky).

To operate on any other points other than the vertex it would be inefficient as the same output can be produced with fewer inputs by moving towards the vertex.

$$\therefore (K/L) \text{ is constant} \Rightarrow d(K/L) = 0$$

$$\Rightarrow \alpha = 0$$

No substitution possible

The mathematical form is

$$q = \min(ak, bl)$$

$$a, b > 0, q_2$$

if $ak < bl$

$$\Rightarrow q = ak$$

The emp. of one more labor

$$\text{is trivial because } MP_L = 0$$

MP results when $ak > bl$.

when $ak > bl$, both inputs are fully utilized. when this happens $k \geq \frac{b}{a}$ and prod takes place at the 1 vertex.

Remember

$$\begin{aligned} f(tk, tl) &= \min(atk, btl) \\ &= t \cdot \min(ak, bl) \\ &= t f(k, l) \end{aligned}$$

$\Rightarrow$ CRS

$$\begin{aligned} ak &< bl \\ tak &< tbl \end{aligned}$$

IRS / DRS can be incorporated by using a non linear transformation of this form $f(k, l)^\gamma$ where $\gamma > 1$ or < 1 .

Case 3 — Cobb Douglas. ($\alpha = 1$)

It provides a middle ground b/w the 2 polar cases discussed before. Cobb Douglas have isoquants which are 'normal' convex to origin.

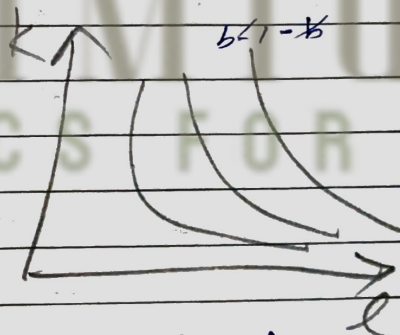
$$q = f(K, L) = A K^a L^b$$

It can be CRS, DRS or IRS depending on values of a & b .

$$\begin{aligned} f(tK, tL) &= A (tK)^a (tL)^b \\ &= t^{a+b} A K^a L^b \\ &= t^{a+b} f(K, L) \end{aligned}$$

If $a+b > 1 \Rightarrow$ CRS $a+b < 1 \Rightarrow$ DRS $a+b > 1 \Rightarrow$ IRS

$$\begin{aligned} \text{RTS} \frac{MP_L}{MP_K} &= \frac{A K^a b L^{b-1}}{A K^{a-1} L^b} \\ &= \frac{b}{a} \left(\frac{K}{L} \right) \end{aligned}$$



$$\ln \text{RTS} = \ln \left(\frac{b}{a} \right) + \ln \left(\frac{K}{L} \right)$$

$$\frac{\partial \ln \text{RTS}}{\partial \ln (K/L)} = 1$$

This prop. is useful in many applications because it is linear.

$$\ln q = \ln A + a \ln K + b \ln L$$

where

a — Elasticity of output w.r.t K
 b — " " " " L

Case 4 CES prod fn

A functional form that incorporates all the 3 case with σ taking any values is called CES prod fn.

$$q = f(K, L) = (K^p + L^p)^{1/p}$$

$$p \leq 1 \quad p \neq 0 \quad \sigma > 0$$

For $\sigma > 1$, the fn exhibits IRS
 $\sigma < 1$ DRS

$$\sigma > 1 \Rightarrow q = (K^p + L^p)^{1/p}$$

$$= t^p K^p + t^p L^p$$

$$= [t^p (K^p + L^p)]^{1/p}$$

$$= t^{p \cdot \frac{1}{p}} (K^p + L^p)^{1/p}$$

$$= t^1 (K^p + L^p)^{1/p}$$

$$= t f(K, L)$$

CRS

for σ

RTS

$$\frac{MP_L}{MP_K}$$

$$= \frac{\frac{1}{p} (K^p + L^p)^{\frac{1}{p}-1} \cdot p L^{p-1}}{\frac{1}{p} (K^p + L^p)^{\frac{1}{p}-1} \cdot p K^{p-1}}$$

$$= \left(\frac{L}{K} \right)^{p-1} = \left(\frac{K}{L} \right)^{1-p}$$

$$\ln RTS = (1-p) \ln\left(\frac{k}{l}\right)$$

$$\frac{\frac{\partial \ln RTS}{\partial k}}{\frac{\partial \ln RTS}{\partial l}} = \frac{\frac{\partial \ln(k/l)}{\partial \ln(k/l)}}{\frac{\partial \ln(k/l)}{\partial \ln(k/l)}} = \frac{1}{1-p} //$$

Alt

Use the formula $\left[\frac{\frac{\partial k/l}{\partial MRTS}}{\frac{\partial MRTS}{k/l}} \right]$ Refer NME-2 note //

Hence linear, fixed prop. & CD cases correspond to $p=1$, $p=-\infty$ and $p=0$.

as $p \rightarrow 0$, CES $\rightarrow$ CD.

ie with CES and $p=0$ the fn converges to CD form
often $q = k^\beta l^{1-\beta}$

Often CES fn is used with distributional weight β , β ($0 \leq \beta \leq 1$) to indicate the relative significance of inputs: -

$$q = f(k, l) = [\beta k^\rho + (1-\beta) l^\rho]^{1/\rho}$$

ex 9.3a.

$$q = f(k, l) = k + l + 2\sqrt{k \cdot l}$$

$$p \cdot f(k, l) = tk + tl + 2\sqrt{t^2 k l}$$

$$= tk + tl + 2t\sqrt{k l}$$

$$= t(k + l + 2\sqrt{k l})$$

$$= t f(k, l)$$

CRS

$$MP_L = 1 + \frac{K}{2\sqrt{K L}}$$

$1 - \frac{1}{2}$

$$= 1 + \frac{K}{\sqrt{K L}} = 1 + (K/L)^{1/2}$$

$$MP_K = 1 + \frac{L}{\sqrt{K L}} = 1 + (K/L)^{-1/2}$$

$$RTS = \frac{1 + (K/L)^{1/2}}{1 + (K/L)^{-1/2}}$$

To find α .

$$k + l + 2\sqrt{k l} = (\sqrt{k} + \sqrt{l})^2$$

$$(k^{0.5} + l^{0.5})^2$$

$$\Rightarrow p = 0.5 \quad \alpha = 1 \Rightarrow \alpha = \frac{1}{1-p} = \frac{1}{1-0.5} = 2$$

Generally

RTS =

$$\frac{1 + \left(\frac{k}{l}\right)^{0.5}}{1 + \left(\frac{l}{k}\right)^{-0.5}}$$

$$\log a + \log ab \\ \log a + \log b$$

$$\ln RTS = \ln \left(1 + \left(\frac{k}{l}\right)^{0.5} \right) - \ln \left(1 + \left(\frac{l}{k}\right)^{-0.5} \right)$$

$$\frac{\partial \ln RTS}{\partial \ln(l/k)}$$

(later)

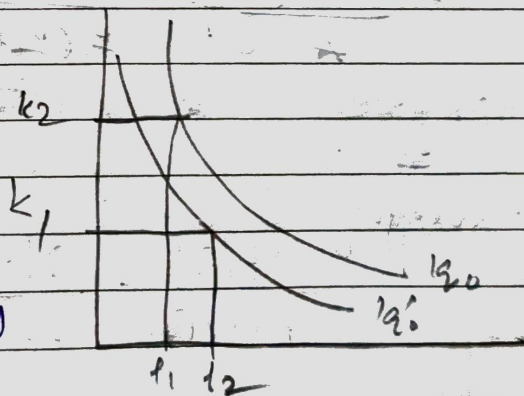
Technical progress

Methods of production improve over time and thus it affects the q_0 . One such progress is shown in the fig where development of superior technique makes the q_0 to shift to q'_0 (towards origin). Now the same level of output can be produced with fewer inputs (For expt read 299 tech progress first para).

However so that

an $\uparrow$ in capitalto k_2 would alsomean reduction in labor along the original q_0 in which case

output per worker $\uparrow$ although there's no real tech progress.



Technical progress.

Let $q = A(t) f(k, l)$.

$$\Rightarrow \frac{q}{A} = f(k, l) \quad (1)$$

Changes in A over time represents technical progress.

$$\Rightarrow A = \frac{q}{f(k, l)}$$

Recall $\frac{dA}{dt} > 0$.

$$\begin{aligned} \frac{dq}{dt} &= \frac{dA}{dt} \cdot f(k, l) + A \cdot \frac{df(k, l)}{dt} \\ &= \frac{dA}{dt} \cdot \frac{q}{A} + \frac{q}{f(k, l)} \cdot \left(\frac{\partial f}{\partial k} \frac{dk}{dt} + \frac{\partial f}{\partial l} \frac{dl}{dt} \right) \end{aligned}$$

÷ by q

$$\begin{aligned} \frac{dA}{dt} \cdot \frac{1}{A} + \frac{1}{f(k, l)} \left[\frac{\partial f}{\partial k} \frac{dk}{dt} + \frac{\partial f}{\partial l} \frac{dl}{dt} \right] \\ = \frac{dA/dt}{A} + \frac{\partial f / \partial k}{f(k, l)} \frac{dk}{dt} + \frac{\partial f / \partial l}{f(k, l)} \frac{dl}{dt} \end{aligned}$$

Rearranging

$$\frac{\partial f / \partial k}{f(k, l)} = \frac{\partial f}{\partial k} \cdot \frac{k}{f(k, l)} \cdot \frac{dk/dt}{k}$$

and $\frac{\partial f / \partial l}{f(k, l)} = \frac{\partial f}{\partial l} \cdot \frac{l}{f(k, l)} \cdot \frac{dl/dt}{l}$

$$\Rightarrow \frac{dq/dt}{q} = \frac{dA/dt}{A} + \frac{\partial f}{\partial k} \cdot \frac{k}{f(k,l)} \cdot \frac{dk/dt}{k} + \frac{\partial f}{\partial l} \cdot \frac{l}{f(k,l)} \cdot \frac{dl/dt}{l}$$

For any variable q , $\frac{dq}{dt}/q$ is the proportional rate of growth of q per unit time

$$\Rightarrow G_q = G_A + \frac{\partial f}{\partial k} \cdot \frac{k}{f(k,l)} \cdot G_k + \frac{\partial f}{\partial l} \cdot \frac{l}{f(k,l)} \cdot G_l$$

Note that $\frac{\partial f}{\partial k} \cdot \frac{k}{f} = \text{elasticity of output w.r.t input } k = e_{q,k}$

$$\text{Similarly } \frac{\partial f}{\partial l} \cdot \frac{l}{f} = e_{q,l}$$

$$\Rightarrow G_q = G_A + e_{q,k} \cdot G_k + e_{q,l} \cdot G_l$$

Rate of growth in output can be broken into sum of 2 components: growth attributed to changes in inputs and other "residual" growth (that is changes in A) that represents technical progress.