PAGE NO.
DATE / /

MME 2.

## Functions of several variables :

### Functions of Two or More Variables :

A function $f$ of two variable $x$ and $y$ with domain $D$ is a rule that assigns a specified number $f(x,y)$ to each point $(x,y)$ in $D$.

A function of $n$ variables $x_1 \dots x_n$ with domain $D$ is a rule that assigns a specified number $f(x_1, \dots x_n)$ to each n-vector $(x_1, \dots x_n)$ in $D$.



### AM, GM and HM :

AM : $\bar{x}_A = \frac{1}{n}(x_1 + x_2 + \dots x_n)$

GM : $\bar{x}_G = \sqrt[n]{x_1 x_2 \dots x_n}$

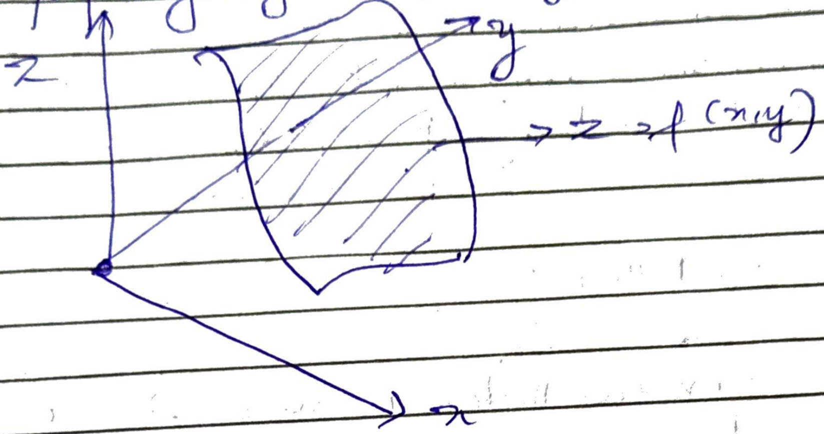
HM : $\bar{x}_H = \dfrac{1}{\frac{1}{n}\left(\frac{1}{x_1} + \frac{1}{x_2} + \dots + \frac{1}{x_n}\right)}$.

Note : $\bar{x}_H \leq \bar{x}_G \leq \bar{x}_A$

HM $\leq$ GM $\leq$ AM

## Graph of a function of Two variables

The graph of $z = f(x,y)$.




**Level curves:**
A level curve also called a contour line represents the set of all combinations of two variables (typically $x$ and $y$) for which a given function $f(x,y)$ equals a constant value $k$. In other words it is the curve formed by solving $f(x,y) = k$.

eg: Indifference curves : $U(x,y) = k$.
Isoquants : $Q(L,k) = k$
Budget line : $P_y \cdot y + P_x \cdot x = M$.

**Tangent planes:**

The tangent plane to $z = f(x,y)$ at the point $(x_0, y_0, z_0)$ with $z_0 = f(x_0, y_0)$ has the equation

$$z - z_0 = f_1'(x_0, y_0)(x - x_0) + f_2'(x_0, y_0)(y - y_0).$$

PAGE NO.
DATE / /

Partial derivatives with many variables.

If $z = f(x_1, x_2, \cdots, x_n)$ then $\frac{\partial f}{\partial x_i}$ is the derivative of $f(x_1, x_2, \cdots, x_n)$ w.r.t. $x_i$ when all other variables $x_j$ $(j \neq i)$ are held constant.

Hessian Matrix.

The Hessian matrix is a square matrix of second order partial derivatives of a scalar valued function. It is used to analyze the curvature of a function and plays a key role in optimization problems.

Definition.

For a function $f(x_1, x_2, \cdots, x_n)$, the Hessian matrix is defined as:

$$H(f) = \begin{bmatrix} \partial^2 f/\partial x_1^2 & \partial^2 f/\partial x_1 \partial x_2 & \cdots & \partial^2 f/\partial x_1 \partial x_n \\ \partial^2 f/\partial x_2 \partial x_1 & \partial^2 f/\partial x_2^2 & \cdots & \partial^2 f/\partial x_2 \partial x_n \\ \vdots & \vdots & \ddots & \vdots \\ \partial^2 f/\partial x_n \partial x_1 & \partial^2 f/\partial x_n \partial x_2 & \cdots & \partial^2 f/\partial x_n^2 \end{bmatrix}$$

- The Hessian is symmetric if $f$ is twice differentiable meaning $\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}$
- It provides information about curvature of the function $f$.

PAGE NO.
DATE / /

Young's Theorem

Suppose that two mth order partial derivatives of the function $f(x_1, x_2, \ldots, x_n)$ involve the same number of differentiation with respect to each of the variables, and are both continuous in an open set S. Then the two partial derivatives are necessarily equal at all points in S.

In particular case of $m=2$,

$$\frac{\partial^2 f}{\partial x_j \partial x_i} = \frac{\partial^2 f}{\partial x_i \partial x_j} \quad (i = 1, 2, \ldots, n;\ j = 1, 2, \ldots n)$$

Quadratic Forms in Two Variables

A ~~general~~ quadratic form of two variables is a function

$$f(x, y) = ax^2 + 2bxy + cy^2$$

Using matrix notation

$$f(x, y) = (x, y)\begin{pmatrix} a & b \\ b & c \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

The second-order partials of f are $f''_{11} = 2a$, $f''_{12} = f''_{21} = 2b$ and $f''_{22} = 2c$.

So Hessian of f is $2\begin{pmatrix} a & b \\ b & c \end{pmatrix}$

The quadratic form $f(x,y) = ax^2 + 2bxy + cy^2$ is

positive definite $\Leftrightarrow a>0,\ c>0$ and $\begin{vmatrix} a & b \\ b & c \end{vmatrix} > 0$

positive semidefinite $\Leftrightarrow a \geq 0,\ c \geq 0$ and $\begin{vmatrix} a & b \\ b & c \end{vmatrix} \geq 0$

negative definite $\Leftrightarrow a<0,\ c<0$ and $\begin{vmatrix} a & b \\ b & c \end{vmatrix} > 0$

negative semidefinite $\Leftrightarrow a \leq 0,\ c \leq 0$ and $\begin{vmatrix} a & b \\ b & c \end{vmatrix} \geq 0$.

indefinite $\Leftrightarrow$ $\begin{vmatrix} a & b \\ b & c \end{vmatrix} < 0$



Recall that $\begin{vmatrix} a & b \\ b & c \end{vmatrix} = ac - b^2$.

Quadratic Forms in many variables:

A general quadratic form in $n$ variables is a function $Q = Q(x_1, \ldots x_n)$ given by the double sum,

$$Q = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} x_i x_j$$

$$= a_{11}x_1^2 + a_{12}x_1x_2 + \cdots + a_{ij}x_ix_j + \cdots \cdots + a_{nn}x_n^2.$$

Suppose we put

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

PAGE NO.
DATE / /

Then it follows from definition of matrix multiplication that

$$Q(x_1, \dots, x_n) = Q(x) = x'Ax.$$

Note that $x_i x_j = x_j x_i$ ⇒ we can write
$a_{ij} x_i x_j + a_{ji} x_j x_i = (a_{ij} + a_{ji}) x_i x_j$.
If we replace $a_{ij}$ and $a_{ji}$ by $\frac{1}{2}(a_{ij} + a_{ji})$ then $a_{ij}$ and $a_{ji}$ become equal without changing $Q(x_1, \dots x_n)$. Thus we can say

$$a_{ij} = a_{ji} \quad (\forall\ i \text{ and } j)$$

meaning matrix A is symmetric

Suppose A is a symmetric matrix, then:

A is positive definite ⇔ all eigenvalues of A are positive
A is positive semidefinite ⇔ all eigenvalues of A are $\geq 0$
A is negative definite ⇔ all eigenvalues of A are negative
A is negative semidefinite ⇔ all eigenvalues of A are $\leq 0$
A is indefinite ⇔ A has at least two eigenvalues with opposite signs.

~~Princ~~ Leading principal minors.

In the context of square matrix, leading principal minors are determinants of the top left submatrices (principal sub matrices) of increasing size.

For a square matrix $A = (a_{ij})$ of $n \times n$ matrix

ie $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$

$$D_k = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{nk} \end{vmatrix} \quad (k = 1, \ldots n)$$

$D_k$ is obtained ~~to~~ from $|A|$ by crossing out the last $n-k$ columns and corresponding $n-k$ rows.

Thus for $k = 1, 2, 3 \ldots n$, the leading principal minors are,

$$a_{11}, \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{3} \end{vmatrix} \cdots \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

Theorem:- Let $A = (a_{ij})_{n \times n}$ be a symmetric matrix with leading principal minors $D_k$ $(k = 1, 2, \ldots n)$.

then

(a) $A$ is positive definite $\Leftrightarrow D_k > 0$ for $k = 1, 2, \ldots n$

(b) $A$ is negative definite $\Leftrightarrow D_k (-1)^k > 0$ for $k = 1, 2, \ldots n$.

(1)

## The Chain Rule.

When $z = F(x, y)$ with $x = f(t)$ and $y = g(t)$ then

$$\frac{dz}{dt} = F_1'(x, y)\frac{dx}{dt} + f_2'(x, y)\frac{dy}{dt}$$

## Directional derivatives

The directional derivatives of $f(x, y)$ at $(x_0, y_0)$ in the direction of the unit vector $(h, k)$ (that is $h^2 + k^2 = 1$) is

$$D_{h,k} f(x_0, y_0) = f_1(x_0, y_0)h + f_2(x_0, y_0)k \quad \text{— (A)}$$

The vector $(f_1(x_0, y_0), f_2(x_0, y_0))$ is called the gradient of $f(x, y)$ at $(x_0, y_0)$.

~~The~~ Note that only when $(h, k)$ has length 1 does the derivative of the directional function have the following convinent property: A move of one unit away from $(x_0, y_0)$ in direction $(h, k)$ changes the value of $f$ by approximately $D_{h,k} f(x_0, y_0)$

(A) says that $D_{h,k}$ is the scalar product of gradient with vector $(h, k)$.

Second directional derivative:

$$D^2_{h,k} f(x_0, y_0) = f_{11}''(x_0, y_0)h^2 + 2f_{12}(x_0, y_0)hk + f_{22}''(x_0, y_0)k^2$$

PAGE NO.
DATE / /

Generalized Chain rule.

When $z = F(x, y)$ with $x = f(t, s)$ and $y = g(t, s)$, then

(a) $\dfrac{\partial z}{\partial t} = F_1'(x, y)\dfrac{\partial x}{\partial t} + F_2'(x, y)\dfrac{\partial y}{\partial t}$.

(b) $\dfrac{\partial z}{\partial s} = F_1'(x, y)\dfrac{\partial x}{\partial s} + F_2'(x, y)\dfrac{\partial y}{\partial s}$.

Generalizing more .....

Suppose that
$z = F(x_1, \dots, x_n)$ with
$x_1 = f_1(t_1, \dots, t_m), \dots,$
$\dots, x_n = f_n(t_1, \dots, t_m)$.



Then the general chain rule is

$$\frac{\partial z}{\partial t_j} = \frac{\partial z}{\partial x_1}\cdot\frac{\partial x_1}{\partial t_j} + \frac{\partial z}{\partial x_2}\cdot\frac{\partial x_2}{\partial t_j} + \dots + \frac{\partial z}{\partial x_n}\cdot\frac{\partial x_n}{\partial t_j}$$

$(j = 1, 2, \dots, m)$

Leibniz's Formula.

Suppose that $f(t, x)$, $a(t)$ and $b(t)$ are differentiable functions and that:

$$F(t) = \int_{a(t)}^{b(t)} f(t, x)\, dx.$$

Then the derivative of F is given by

$$F'(t) = f(t, b(t))\, b'(t) - f(t, a(t))\, a'(t) + \int_{a(t)}^{b(t)} \frac{\partial f(t,x)}{\partial t}\, dx$$

Derivatives of functions defined implicitly

$$F(x,y) = c$$

$$\Rightarrow \frac{dy}{dx} = -\frac{F_1'(x,y)}{F_2'(x,y)}$$

$$(F_2'(x,y) \neq 0)$$

$$F(x,y) = c \Rightarrow y'' = \frac{d^2y}{dx^2} = \frac{1}{(F_2')^3} \begin{vmatrix} 0 & F_1' & F_2' \\ F_1' & F_{11}'' & F_{12}'' \\ F_2' & F_{21}'' & F_{22}'' \end{vmatrix}$$

valid for $F_2' \neq 0$.

More on implicit differentiation

Consider $F(x,y,z) = c$ where c is a constant

with $z = f(x,y)$

then

$$z_x' = \frac{-F_x'}{F_z'}, \quad z_y' = \frac{-F_y'}{F_z'}$$

$$(F_z' \neq 0)$$

General case

Suppose $z$ is defined implicitly as a differentiable function of the $n$ variables $x_1, \ldots, x_n$ by the equation

$$F(x_1, x_2, \ldots, x_n, z) = c \quad ; \text{ then}$$

$$\frac{\partial z}{\partial x_i} = -\frac{\partial F / \partial x_i}{\partial F / \partial z}$$

$$(i = 1, 2, \ldots, n)$$

Implicit Function Theorem.

Let $F : \mathbb{R}^{n+m} \rightarrow \mathbb{R}^m$ be a continuously differentiable function, where $F(x, y) = 0$ implicitly defines $y$ as a function of $x$ near a point $(x_0, y_0)$ with $F(x_0, y_0) = 0$ and the Jacobian matrix of $F$ w.r.t $y$ denoted as $\frac{\partial F}{\partial y}$ is invertible at $(x_0, y_0)$, then there exists a neighborhood around $x_0$ where $y$ can be expressed as a differentiable function of $x$, denoted $y = g(x)$ such that:

$$F(x, g(x)) = 0.$$

Simply it means if a function $F(x, y)$ implicitly defines $y$ in terms of $x$ and satisfies certain regularity conditions (eg: a non zero determinant of Jacobian w.r.t $y$) then $y$ can be locally expressed as an explicit function $y = g(x)$.

PAGE NO

DATE / /

## Partial elasticities.

If $z = f(x_1, x_2, \dots, x_n)$, we define partial elasticity of $z$ (or of $f$) w.r.t $x_i$ as the elasticity of $z$ w.r.t $x_i$ when all other variables are held constant. Thus,

$$El_i z = \frac{x_i}{f(x_1, x_2, \dots, x_n)} \cdot \frac{\partial f(x_1, x_2 \dots x_n)}{\partial x_i}$$

$$= \frac{x_i}{z} \cdot \frac{\partial z}{\partial x_i}$$

## Elasticities of Composite Functions.

$$z = F(x_1, \dots x_n), \quad x_i = f_i(t_1, \dots, t_m) \quad i = 1, \dots, n$$

$$\Rightarrow El_{t_j} z = \sum_{i=1}^{n} El_i F(x_1, \dots, x_n) El_j x_i$$

## Elasticity of Substitution.

The marginal rate of substitution b/w $y$ and $x$ is

$$R_{yx} = \frac{f'_1(x, y)}{f'_2(x, y)}$$

When $F(x, y) = c$, the elasticity of substitution b/w $y$ and $x$ is

$$\sigma_{yx} = El_{R_{yx}} \left(\frac{y}{x}\right).$$

## Homogeneous Functions of Two Variables :

The function f of two variables x and y defined in a domain D is said to be homogeneous of degree k if for all (x,y) in D,

$$f(tx, ty) = t^k f(x,y) \quad \forall\ t>0.$$

Properties of homogeneity of functions of two variables

> Eulers theorem

$f(x,y)$ is homogenous of degree k

$$\iff x\frac{\partial f}{\partial x} + y\cdot\frac{\partial f}{\partial y} = k\cdot f(x,y)$$

> $f_1'(x,y)$ and $f_2'(x,y)$ are homogeneous of degree k-1

> $f(x,y) = x^k f\left(1, \frac{y}{x}\right) = y^k f(x/y, 1)$ $(x,y>0)$

> $x^2 f_{11}''(x,y) + 2xy f_{12}''(x,y) + y^2 f_{22}''(x,y) = k(k-1) f(x,y)$

Note :- It is often said that graph of a homogeneous function of degree 1 is generated by straight lines through the origin.

## General Homogeneous and Homothetic Functions

Suppose that f is a function of n variables defined in domain D. Suppose that whenever $(x_1, x_2, \dots, x_n) \in D$ and $t>0$, then $(tx_1, tx_2, \dots, tx_n)$ also lies in D. (A set D with this property is called a cone) We say that f is homogeneous of degree k on D if

$$f(tx_1, tx_2, \dots, tx_n) = t^k f(x_1, x_2, \dots, x_n)$$
$$(\forall\ t>0)$$

PAGE NO.
DATE / /

## Eulers Theorem (general)

Suppose $f$ is a function of $n$ variables with continous partial derivatives in an open domain $D$, where $t>0$ and $(x_1, x_2, \dots, x_n) \in D$ imply $(tx_1, tx_2 \dots tx_n) \in D$. Then $f$ is homogeneous of degree $k$ in $D$ iff:

$$\sum_{i=1}^{n} x_i f_i'(x_1, x_2, \dots, x_n) = k f(x_1, x_2, \dots, x_n)$$
$$\forall (x_1, \dots x_n) \in D.$$

## Homothetic Functions:

Let $f$ be a function of $n$ variables $X = (x_1, \dots, x_n)$ defined in a cone $K$. Then $f$ is called homothetic provided that

$$x, y \in K,\ f(x) = f(y),\ t>0 \Rightarrow f(tx) = f(ty)$$

A homogeneous function $f$ of any degree $k$ is homothetic always (not vice versa). For $f$ to be homothetic it must be strictly increasing as well along with it being homogeneous.