

Conditional convergence

This principle holds for the world economy on a conditional basis. (Read footnote on pg 70)

5

Engine of Growth

Now we will be discussing growth models with technology being endogenous. The model that we will be studying is by Paul Romer.

Romer Model

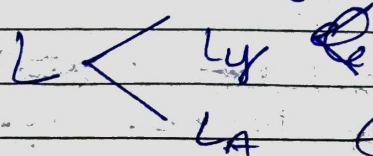
Like the Solow Model, there are two main elements in the Romer model of endogenous technological change: an equation describing the production function and a set of equations describing how inputs for the production function evolve over time.

Production function

$$Y = K^{\alpha} (AL_y)^{1-\alpha} \quad 0 < \alpha < 1$$

K - Capital
A - Tech

Y - output
L_y - Labor



$$s_R = \frac{L_A}{L}$$

(s_R) - (ideas)
population

For given level of technology A , the production function exhibits C.R.S. However, when we recognize that ideas (A) are also an input into prod., then there are IRS.

The accumulation equation of capital and labour as that of Solow Model.

$$\dot{K} = sY - dK$$

Labour which is equivalent to population grows exponentially @ n .

$$\frac{\dot{L}}{L} = n$$

$$L(t) = L(0)e^{nt}$$

In the Solow model $\frac{\dot{A}}{A} = g$, a constant. In this model, that's not the case.

As to Romer model $A(t)$ is the stock of knowledge or the number of ideas that have been invented over the course of time until time period t .

Then $\dot{A}$ is the no. of new ideas produced at any given point in time.

In the simplest version, $\dot{A}$ equals the number of people attempting to discover ideas (L_A), multiplied by the rate

at which they discover new ideas.

i.e. $A = \bar{S} \cdot L_A$ [A/L_A is a net fn of L_A]

The rate at which researchers discover new ideas might be a constant. Or it might be positive in the sense invention of ideas in the past helps in raising the productivity of researchers in present. Or it can have no effect as obvious ideas are discovered first and subsequent ideas are increasingly difficult to discover.

This in the first case $\bar{S}$ would be some constant time A . In the second case $\bar{S}$ is a $\downarrow$ fn of A and in the last case it is an $\uparrow$ fn of A .

The case where $\bar{S} = \bar{S} A^\phi$, $\phi > 0$ — standing on shoulders effect

$\bar{S} = \bar{S} A^\phi$, $\phi < 0$ — fishing out effect

Here we consider $\phi > 0$.

It is also possible that the avg productivity of research depends on the number of people searching for new ideas at any point in time. For example duplication of effort is more likely when there are more persons engaged in research.

one way of modelling this possibility is to use L_A^λ instead of L_A where $0 < \lambda < 1$. (stepping on toes effect)

thus $\dot{A} = \delta L_A^\lambda A^\phi \bar{s}$

For reasons that will become clear, we will assume that $\phi < 1$.

Now we know :- $\dot{A} = \delta L_A^\lambda A^\phi$

$$\dot{K} = s_K Y - dK$$

$$\frac{\dot{L}}{L} = n$$

$$Y = K^\alpha (A L_Y)^{1-\alpha}$$

$$0 < \alpha < 1$$

$$0 < \phi < 1$$

$$0 < \lambda < 1$$

Consider the eqn $\dot{A} = \delta L_A^\lambda A^\phi$.

Individual researchers being small relative to the economy as a whole, take $\bar{s}$ i.e. $\bar{s} = \delta A^\phi$ as given and see CAS to research. But this clearly varies on an aggr. level.

Note: - * The exact expression for $\bar{s}$ incorporating both duplications and knowledge spillovers is $\bar{s} = \delta L_A^{\lambda(\phi-1)}$.

Also note $\lambda < 1$ may reflect an externality associated with duplication.

Two assumptions

> A const fraction of output is invested in capital

$$\rightarrow \underline{L = L_Y + L_A} \quad \text{with} \quad L_A = sR \cdot L$$

It means sR proportion of labor force engages in R&D to produce new ideas whereas remaining $1 - sR$ produces output.

> We assume the economy starts out with K_0, L_0 and A_0 (units of inputs - endowments).

Growth in the Romer model

$$\dot{A} = \delta L_A^\lambda A^\phi$$

$$g_A = \frac{\dot{A}}{A} = \delta L_A^\lambda A^{\phi-1}$$

$$\ln\left(\frac{\dot{A}}{A}\right) = \ln \delta + \lambda \ln L_A + (\phi-1) \ln A$$

diff w.r.t t .

$$g_A = 0 + \lambda \cdot \frac{\dot{L}_A}{L_A} + (\phi-1) g_A$$

$$\text{as } \dot{L}_A/L_A = n \text{ and } \dot{A}/A = g_A$$

Along a bal growth path, the growth rate of
 Residues must be equal to the growth rate of
 capital. If it were higher, no of generations
 would eventually exceed the population.

then $\frac{\dot{L}}{L} = n$ as we said
 then how is $\frac{\dot{L}_A}{L_A} = n$?

and $L = L_Y + L_A$

$L_A = s_R L$

$\ln L_A = \ln s_R + \ln L$

$\frac{\dot{L}_A}{L_A} = 0 + \frac{\dot{L}}{L}$
 $= n$

So we know

~~$\dot{g}_A = \delta n + (\phi - 1) g_A$~~

$\dot{g}_A = \delta n + (\phi - 1) g_A$ (1)

Along Bal Growth path $\dot{g}_A = 0$ (now?)

Provided a constant fraction of the population
 is employed producing ideas, the
 model follows neo-classical model.
 in predicting that all per capita
 growth is due to tech progress.

$\hat{y} = \hat{k} = \hat{g}_A$

ie capital output ratio must be constant
 at BGP.

$$\frac{\dot{A}}{A} = g_A = \frac{\delta A^\alpha}{A^{1-\phi}} \text{ in this model.}$$

so g_A is constant iff both N and D are equal.

$$\text{i.e. } \ln g_A = \ln \delta.$$

ie after taking log and diff. (eqn*) we must have to equate 1 to 0.
we must be

$$\Rightarrow \frac{d \ln g_A}{dt} = 0.$$

$$\Rightarrow \left| g_A = \frac{\delta n}{1-\phi} \right| \text{ along BGP}$$

Thus the long run growth rate of this economy is determined by the parameters of the prod fn for ideas (δ and ϕ) and the rate of growth of researchers (n).

ie $\uparrow$ in δ means lesser duplication problem and ^{more} ideas created by an individual one now and hence more new ideas are generated ($\delta \geq 1 \Rightarrow$ no duplication at all)

ϕ talks about the productivity of researchers (knowledge spillovers). As $\phi \uparrow$, it implies the productivity of all the researchers have $\uparrow$ and therefore ideas growth rises.

Before moving on to the effect of n i.e. pop'n growth let us see a number behind this eqn through a special case where $\delta = 1$ and $\phi = 0$.

This implies $\dot{A} = \delta L_A$ and $g_A = n$.

Take L_A to be constant and δ is already a constant; this economy generates a constant number of new ideas δL_A each period. To be more concrete suppose $\delta L_A = 100$; the economy begins with some stock of ideas A_0 , generated. Initially the 100 new ideas per period may be a large fraction of existing stock A_0 . Over time, the stock grows and the 100 new ideas become a smaller and smaller fraction of the existing stock.

The tech progress never ceases. It is the growth rate of the stock of ideas fall over time.

$$\dot{A} = \delta L_A$$

$$= \delta s r L$$

(with $\gamma > 1$ & $\phi > 0$)

$$\frac{\dot{A}}{A} = \delta s r \frac{L}{A}$$

$s r$ is const

$$\Rightarrow \frac{\dot{L}}{L} \Rightarrow \frac{\dot{A}}{A} \downarrow$$

and its growth

Thus L is important or else $\dot{A}$ eventually reaches zero ^{in the way} as we mentioned earlier

Thus (without labor) a constant research effort cannot continue the proportional increases in the stock of ideas needed to generate long run growth.

Note There is one special case in which a constant research effort can sustain long run growth when $\gamma = 1$ and $\phi = 1$.

$$\Rightarrow \dot{A} = \delta L_A^\gamma A^\phi \text{ becomes}$$

$$\dot{A} = \delta L_A A$$

$$\Rightarrow \frac{\dot{A}}{A} = \delta L_A$$

$$\Rightarrow \delta s r L$$

ie with cost γ also one have constant growth of A

Here another assumption - made by Romer is that productivity of research is proportional to existing stock of ideas.
i.e. $\dot{g} = \delta A$ so that productivity of research goes over time even if no of researchers is const. (read 106 and para)

But empirical evidence suggests that $\phi \neq 1$ and also $\phi \neq 1$. Thus $\phi < 1$.

Just like the previous models, even with endogenous technology, the long run growth rate cannot be manipulated by policy maker using conventional policies.

Growth effects Vs level effects -

An increase in the R&D share

Take $\phi = 0$ and $\alpha = 1$

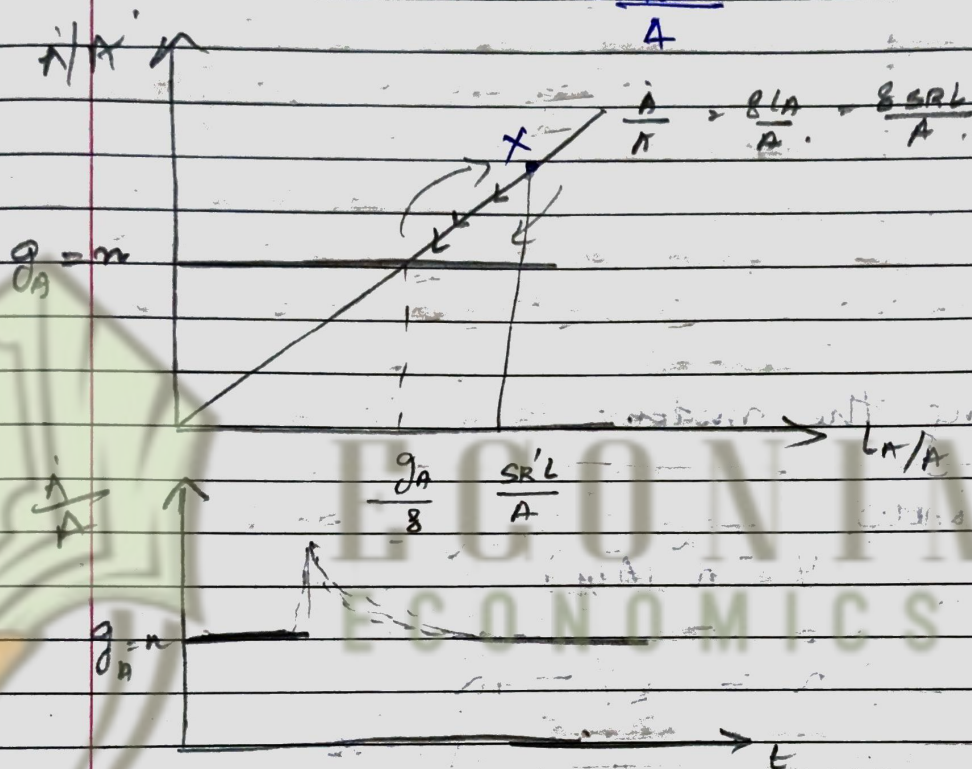
Along BGP,

$$\frac{\dot{A}}{A} = n$$

Now SR jumps $\Rightarrow \frac{\dot{A}}{A} > n$. But it gradually comes back to n .

We know $\frac{\dot{A}}{A} = \frac{8 L_A}{A}$

$$= \frac{8 S R L}{A}$$

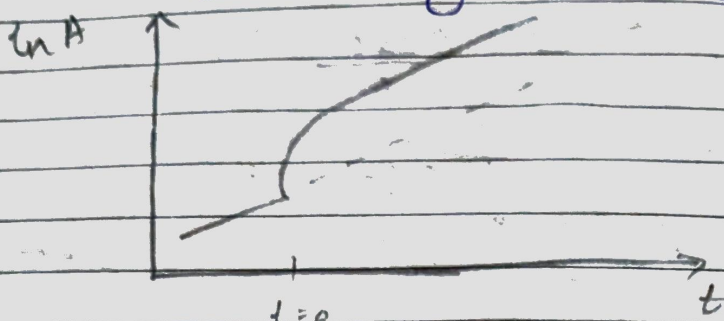


In steady state the economy grows along a balanced growth path at the rate of technological progress g_A which equal rate of growth of popln (due to assumption of $\phi \delta$).
 Let $SR \uparrow$ at time t_0 . With popln L_0 , the no. of researchers $\uparrow$ as $SR \uparrow \Rightarrow \frac{L_A}{A}$ jumps to a higher level.

This additional research produces an $\uparrow$ ed number of new ideas & extracted via X .

At $t = X$, $\frac{\dot{A}}{A} > n$. Over time the ratio declines until the economy comes back to n .

The level of technology is permanently higher



Solving the model.

We know

$$Y = K^{\alpha} (A L)^{1-\alpha}$$

$$\dot{K} = s_K Y - dK$$

$$\frac{Y}{A L} = \left(\frac{K}{A L} \right)^{\alpha}$$

$$\tilde{y} = \tilde{k}^{\alpha}$$

$$\dot{\tilde{k}} = s_K \cdot \tilde{k}^{\alpha} - (n + g_A + d) \tilde{k}$$

$$\tilde{k}^* = \left[\frac{s_K}{n + g_A + d} \right]^{\frac{1}{1-\alpha}}$$

$$\Rightarrow \tilde{y}^* = \left[\frac{s_K}{n + g_A + d} \right]^{\frac{\alpha}{1-\alpha}}$$

$$\Rightarrow \frac{Y}{A L} = \left[\frac{s_K}{n + g_A + d} \right]^{\frac{\alpha}{1-\alpha}}$$

$$\frac{y}{A(1-sr)L} = \left[\frac{s_k}{n+s_A+d} \right]^{\frac{\alpha}{1-\alpha}}$$

$$y^* = \frac{y}{L} = \left[\frac{s_k}{n+s_A+d} \right]^{\frac{\alpha}{1-\alpha}} (1-sr)A$$

$$g_A = \frac{s_{sr} L}{A}$$

$$A = \frac{s_{sr} L}{g_A}$$

$$\frac{y}{L} = \left[\frac{s_k}{n+s_A+d} \right]^{\frac{\alpha}{1-\alpha}} sr(1-sr) \frac{s_{sr} L}{s_A}$$

$$y(L) = \left[\frac{s_k}{n+g_A+d} \right]^{\frac{\alpha}{1-\alpha}} sr(1-sr) \frac{s_{sr}}{s_A} \cdot L(L)$$

The model exhibits a scale effect in levels i.e. a larger world economy will be a richer world economy. This scale effect arises fundamentally from the non-rival nature of ideas. A larger economy provides a larger market for ideas (add effect). In addition a more populous world economy simply has more potential creators (a SS effect).

Note that sr comes in 2 ways. - no effect as more research means less ly and second the effect $\uparrow sr \Rightarrow \uparrow$ researchers $\Rightarrow \uparrow$ productivity of economy.

Economics of the model

The Romer economy consists of 3 sectors: a final goods sector, an intermediate goods sector and a research sector.

The research sector creates new ideas which take the form of capital goods which then is sold (the end right) to an intermediate goods firms to produce a specific capital good. The int goods firm as a monopolist, manufactures the capital good and sells it to the final goods sector which produces output.

Final goods sector

It consists of a large number of perfectly competitive firms that combine labour and capital to produce a homogeneous output good Y .

Prod fn:

$$Y = L_y^{1-\alpha} \sum_{j=1}^A x_j^{\alpha}$$

Output Y is produced using labor L_y and a number of diff. capital goods x_j which we will also call intermediate goods.

At any time A measures the no of capital goods that are available to be used in the final goods sector, and firms in the final goods sector take this no as given.

Prod. fn can be rewritten as:

$$Y = L y^{1-\alpha} (\pi_1^\alpha + \dots + \pi_A^\alpha)$$

for a given A , the production function exhibits CRS.

If A is ↑ then IRS.

For technical reasons, we replace the summation with integral.

$$Y = L y^{1-\alpha} \int_0^A \pi_j^\alpha dj$$

set price of final good $Y = 1$.

" " labor $y = w$.

j^{th} capital good $= p_j$

Revenue

Optimization

π • $p (= 1)$ - cost

$$\max_{y, \pi_j} \underbrace{L y^{1-\alpha} \int_0^A \pi_j^\alpha dj}_{\text{Revenue}} - \underbrace{w L y - \int_0^A p_j \pi_j dj}_{\text{Cost}}$$

First order conditions

$$\frac{\partial \pi}{\partial y} (1-\alpha) y^{-\alpha} \cdot \int_0^A \pi_j^{\alpha} dj - \omega = 0$$

$$\Rightarrow (1-\alpha) y^{-\alpha} \cdot \int_0^A \pi_j^{\alpha} dj - \omega = 0$$

$$(1-\alpha) y^{-\alpha} \cdot \int_0^A \pi_j^{\alpha} dj = \omega$$

$$(1-\alpha) \cdot y^{-\alpha} \cdot \int_0^A \pi_j^{\alpha} dj = \omega$$

and $\div$ by $y^{-\alpha}$

$$\Rightarrow \frac{(1-\alpha)}{y^{-\alpha}} \cdot \int_0^A \pi_j^{\alpha} dj = \omega$$

$$\frac{(1-\alpha)}{y^{-\alpha}} Y = \omega$$

$$\Rightarrow \omega = \frac{(1-\alpha) Y}{y^{-\alpha}} \quad \text{--- (1)}$$

Now

$$\frac{\partial \pi}{\partial \pi_j} = (1-\alpha) y^{-\alpha} \alpha \pi_j^{\alpha-1} - p_j = 0$$

$$\Rightarrow p_j = \alpha y^{-\alpha} \cdot \pi_j^{\alpha-1}$$

Eqn ① and ② represents the demand for labor and capital goods in the final goods sector.

Intermediate goods sector:

The intermediate goods sector consists of monopolists who produce the capital goods that are sold to the final goods sector. These firms gain their monopoly power by purchasing a design for a specific capital goods from the research sector. Because of patent protection, only one firm manufactures each capital good.

x_j is produced; cost = z ; price = $P_j(x_j)$.

⇒ Optimization is

$$\max_{x_j} P_j(x_j) \cdot x_j - z \cdot x_j$$

FOC

$$\frac{\partial \pi}{\partial x_j} = P_j(x_j) + x_j P_j'(x_j) - z = 0$$

Removing subscripts for simplicity

$$\text{FOC: } P(x) + P'(x) \cdot x - z = 0$$

÷ throughout by $p(n)$

$$\Rightarrow \frac{p(n) \cdot n}{p(n)} + 1 = \frac{\gamma}{p(n)}$$

add elasticity

$$\Rightarrow p(n) = \frac{1}{1 + \frac{p(n) \cdot n}{p}}$$

the elasticity $\frac{p(n) \cdot n}{p}$ can be calculated from the ed one in equation 5.12
 $p_j = \alpha y^{1-\alpha} n_j^{\alpha-1}$

It is equal to $\alpha - 1$. (⊗) How?

$$\Rightarrow p(n) = \frac{1}{1 + \alpha - 1} \cdot \gamma$$

$$p(n) = \frac{\gamma}{\alpha}$$

So the intermediate goods firm charges a price that is simply a markup over marginal cost, α .

$$p_j^n = \alpha L y^{1-\alpha} n_j^{\alpha-1}$$

$$p_j'(n_j) = \alpha L y^{1-\alpha} (\alpha-1) n_j^{\alpha-2}$$

$$= \frac{\alpha-1}{n_j} p_j$$

$$\Rightarrow \frac{p_j'(n_j) \cdot n_j}{p_j} = \alpha - 1$$

$$\pi = p n - r n$$

$$= p n - \alpha p n$$

$$\pi = (1-\alpha) p n$$

④ All capital goods sell for same price. $\therefore$ the dd fn of n and is same. Each capital good is employed by the final good at same amt $n_j = n$. $\therefore$ Each firm earns the same π as well.

Total demand for capital from the intermediate goods firms must equal the total capital stock in the economy:

$$\int_0^A n_j dj = K$$

Since the capital goods are used in the same amount n to $K = n \int_0^A dj = K \cdot n (A-0)$

$$\Rightarrow n = \frac{K}{A} \quad \text{④}$$

$\Rightarrow$ Prod fn can be rewritten as

$$Y = A L y^{1-\alpha} n^{\alpha}$$

$$\text{put } \pi = \frac{k}{A}$$

$$\Rightarrow Y = A \gamma^{1-\alpha} A^{-\alpha} K^{\alpha} \\ = K^{\alpha} (A \gamma)^{1-\alpha}$$

$$P_{\pi} = \alpha \gamma^{1-\alpha} \pi^{\alpha} \\ = \alpha \gamma^{1-\alpha} \left(\frac{k}{A} \right)^{\alpha}$$

$\times \alpha \div$ by A

$$\Rightarrow P_{\pi} = \alpha \frac{\gamma}{A}$$

$$\Rightarrow \pi = \alpha (1-\alpha) \gamma / A //$$

Research sector

Ideas in this model are designs for new capital goods, which transform raw capital into a ^{unit of} new capital good.

When a new design is discovered, the inventor secures a patent from the government for the exclusive right to produce the new capital good. The inventor sells the patent to int good firms and uses the proceeds to either consume or save.

How much will a potential bidder be willing to pay? The answer is: the PDV of the profits to be earned by an int. goods firm. Any less and someone'd bid higher and none of its own.

Let P_A be the price of a new design. How does P_A change over time?

Arbitrage argument (read last para of pg 115)

$$\sigma P_A = \pi + \dot{P}_A$$

$$\sigma = \frac{\pi}{P_A} + \frac{\dot{P}_A}{P_A}$$

Along a BGP, σ is constant (it'll be the price at which the supply of capital is equal to demand for capital and will be proportional to $\frac{Y}{K}$).

$\therefore \pi/P_A$ must be constant means they have to grow at the same rate (i.e. n).

This arbitrage implies $P_A = \frac{\pi}{\sigma - n}$.

Solving the model

Note that prod fn are CRS to K and L but IRS once we put A (ideas) as an input.

A appears in the int. goods mkt where which is a monopoly and capital goods sold at a price that is greater than MC.

Research sector is monopolistic competition. There are no economic profits; all rents compensate some factor input.

Let us assume σ is constant, but determined endogenously by the model.

Remember individuals in this simplified model are indifferent b/w working in the final goods sector and working in research sector.

Labor working in the final goods sector earns a wage that is equal to its marginal product in that sector.

$$\text{ie } w_y = (1-\alpha) \frac{Y}{L_y}$$

Researchers earn a wage based on the value of the designs.

Assume that researchers take their productivity in the research sector $\bar{s}$ as given.

$\therefore$ the wage earned by labor in the research sector is equal to its marginal product $\bar{s}$, multiplied by the value of new ideas created P_A :

$$w_R = \bar{s} P_A$$

And as noted earlier $w_Y = w_R$.

$$\text{ie } (1-\alpha) \frac{Y}{L_Y} = \bar{s} P_A$$

$$(1-\alpha) \frac{Y}{L_Y} = \bar{s} \frac{\pi}{2-n} \quad \left[P_A = \frac{\pi}{2-n} \right]$$

$$\text{Also } \pi = \alpha(1-\alpha) \frac{Y}{A}$$

$\Rightarrow$ eqn becomes

$$\cancel{(1-\alpha)} \cdot \cancel{\frac{Y}{L_Y}} = \frac{\bar{s}}{2-n} \alpha \cancel{(1-\alpha)} \frac{\cancel{Y}}{A}$$

$$\frac{\alpha \bar{g}}{(\alpha - n)} A = \frac{1}{L_y}$$

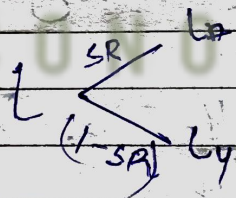
We know $\frac{A}{A} = g_A = \frac{g_A}{A}$

with $\hat{A} = 1$
 $\phi = 0$

$$\Rightarrow \frac{\bar{g}}{A} = \frac{g_A}{L_A}$$

$\Rightarrow$ $\frac{\alpha g_A}{\alpha - n} = \frac{L_A}{L_y}$

We know



$$\Rightarrow \frac{\alpha g_A}{\alpha - n} = \frac{s_R}{1 - s_R}$$

Solving for s_R .

$$s_R = \frac{1}{1 + \frac{\alpha - n}{\alpha g_A}}$$

Note

$\uparrow g_A \Rightarrow \uparrow \text{Growth (as } \uparrow s_A)$

$\uparrow (s - n) \Rightarrow (\downarrow s_A)$

$$K^{\alpha} (A L)^{1-\alpha}$$

$$\frac{\partial}{\partial K} (K^{\alpha} (A L)^{1-\alpha}) = \alpha K^{\alpha-1} (A L)^{1-\alpha}$$

also note that

$$r = \frac{\alpha^2 Y}{K}$$

We know marginal product of capital $= \frac{\alpha Y}{K}$

Hence here $r < MP_K$

In the Solow model, prod fn exhibits CRS and hence all factors are paid their marginal product

is $r = \alpha Y$ and $w = (1-\alpha) Y$ and thus $rK + wL = K Y$. There is no output in Solow economy to compensate individuals for their effort in creating new A.

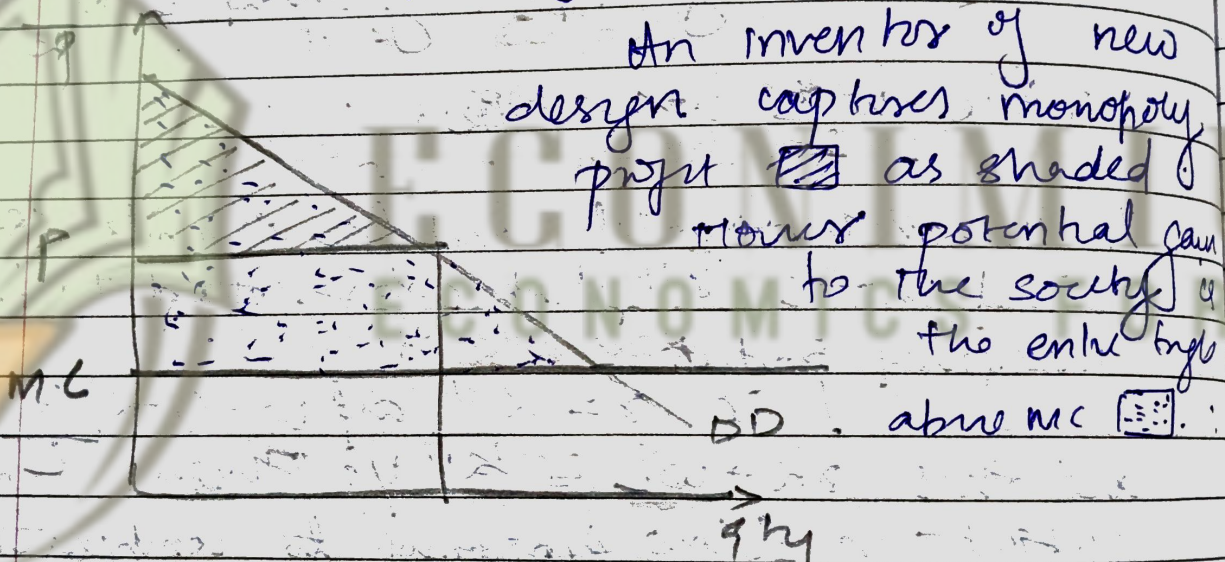
This necessitates the need for imperfect competition in the model. The production in Romer model exhibits IRS and all factors can't be paid their marginal products. Here capital is paid less than its marginal product.

Optimal

Optimal R&D

$\phi > 0 \rightarrow$ Stepping on toes effect } Read.
 $\lambda < 1 \rightarrow$ Standing on shoulders effect

Consumer Surplus effect -



The incentive to innovate, the monopoly profit is less than gain to society, and this effect tends to generate too little innovation at par.